

Painlevé equations and the middle convolution

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Abstract

We use the middle convolution to obtain some old and new algebraic solutions of the Painlevé VI equations.

1 Introduction

A description of all irreducible and physically rigid local systems on the punctured affine line was given by Katz [7]. The main tool here-fore is a middle convolution functor on the category of perverse sheaves, loc. cit., Chap. 5. This functor is denoted by MC_χ , for χ a one-dimensional representation of $\pi_1(\mathbb{G}_m)$. It preserves important properties of local systems (resp. perverse sheaves) like the index of rigidity and irreducibility, but in general, MC_χ changes the rank and the monodromy group.

In [3], the authors give a purely algebraic analogon of the functor MC_χ (the construction is reviewed in Section 2). This functor is a functor of the category of modules of the free group F_r on r generators to itself. It depends on a scalar $\lambda \in \mathbb{C}^\times$ and is denoted by MC_λ . Clearly, MC_λ can be viewed as a transformation which sends an r -tuple $(M_r, \dots, M_1) \in \mathrm{GL}_m^r$ to another r -tuple $MC_\lambda(M_r, \dots, M_1) \in \mathrm{GL}_n^r$, where usually $m \neq n$. It is shown in [3], that this transformation commutes with the action of the Artin braid group (up to overall conjugation in GL_n). It is this property, which makes the middle convolution functor MC_λ useful for the study of the Painlevé equations.

The sixth Painlevé equation $\mathrm{PVI}_{\alpha,\beta,\gamma,\delta}$ is a nonlinear differential equation, depending on 4 parameters $\alpha, \beta, \gamma, \delta \in \mathbb{C}$:

$$y'' = \frac{1}{2} \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t} \right) (y')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t} \right) y' + \frac{y(y-1)(y-t)}{t^2(t-1)^2} \left(\alpha + \frac{\beta t}{y^2} + \frac{\gamma(t-1)}{(y-1)^2} + \frac{\delta t(t-1)}{(y-t)^2} \right).$$

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It is well known, that PVI arises from isomonodromic deformations (Schlesinger equations), see [6] and [5]. This approach is recalled in Section 3. Using this approach, one can show that finite braid group orbits of triples in $\mathrm{SL}_2(\mathbb{C})$ give rise to solutions of PVI with finite branching. It is expected, that these solutions are always algebraic, see [1], P. 1012.

In this paper, we give a convolution approach to the “Klein solution” of PVI, studied by Boalch [1], [2]. Originally, this solution is constructed using a braid invariant map φ from triples of pseudo reflections in $\mathrm{GL}_3(\mathbb{C})$ to triples in $\mathrm{SL}_2(\mathbb{C})$. This map is applied to a generating triple of reflections of the complex reflection group associated to Klein’s simple group of order 168. Since the order of this group is finite, the generating triple gives rise to a finite braid orbit, from which the existence of an algebraic solution of PVI is derived.

We show that the transformation φ is obtained from a suitable middle convolution and scaling. Thus a new approach to the Klein solution is given. Then it is shown, how to obtain new algebraic solutions for PVI by starting with tuples in $\mathrm{GL}_2(\mathbb{C})$. Moreover, we give an example of an algebraic solution which arises from a rational pullback but cannot be obtained from the middle convolution. One may ask whether all algebraic solutions of PVI arise either by middle convolution and scaling or from a rational pullback.

The organization of the paper is as follows: In Section 2, we recall the definition and the basic facts of MC_λ . Section 3 is concerned with some well known facts about Painlevé equations, isomonodromic deformations and braid orbits. Section 4 recalls Boalch’s map φ and gives the connection to MC_λ . Finally, in Section 5 we give some new examples of algebraic solutions of P_{VI} , using the convolution.

2 The middle convolution functor MC_λ

In this section, we recall the algebraic construction of the convolution functor MC_λ , defined in [3] and [4].

We will use the following notations and conventions throughout the paper: Let K be a field and G a group. The category of finite dimensional left- G -modules is denoted by $\mathrm{Mod}(K[G])$. Mostly, we do not distinguish notationally between an element of $\mathrm{Mod}(K[G])$ and its underlying vector space. Let F_r denote the free group on r generators f_1, \dots, f_r . An element in $\mathrm{Mod}(K[F_r])$ is viewed as a pair (\mathbf{M}, V) , where V is a vector space over K and $\mathbf{M} = (M_r, \dots, M_1)$ is an element of $\mathrm{GL}(V)^r$ such that f_i acts on V via M_i , $i = 1, \dots, r$. For $(\mathbf{M}, V) \in \mathrm{Mod}(K[F_r])$, where $\mathbf{M} = (M_r, \dots, M_1) \in \mathrm{GL}(V)^r$, and $\lambda \in K^\times$ one can construct an element $(C_\lambda(\mathbf{M}), V^r) \in \mathrm{Mod}(K[F_r])$, $C_\lambda(\mathbf{M}) = (N_r, \dots, N_1) \in \mathrm{GL}(V^r)^r$, as follows: For $k = 1, \dots, r$, N_k maps a vector

$(v_1, \dots, v_r)^{\text{tr}} \in V^r$ to

$$\begin{pmatrix} 1 & 0 & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ (M_1 - 1) & \dots & (M_{k-1} - 1) & \lambda M_k & \lambda(M_{k+1} - 1) & \dots & \lambda(M_r - 1) \\ & & & 1 & & & \\ & & & & \ddots & & \\ 0 & & & & 0 & & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ v_r \end{pmatrix}.$$

We set $C_\lambda(\mathbf{M}) := (N_r, \dots, N_1)$. There are the following $\langle N_1, \dots, N_r \rangle$ -invariant subspaces of V^r :

$$\mathcal{K}_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \ker(M_k - 1) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (k\text{-th entry}), \quad k = 1, \dots, r,$$

and

$$\mathcal{L} = \cap_{k=1}^r \ker(N_k - 1) = \ker(N_1 \cdots N_r - 1).$$

Let $\mathcal{K} := \oplus_{i=1}^r \mathcal{K}_i$.

Definition 2.1 Let $MC_\lambda(\mathbf{M}) := (\tilde{N}_r, \dots, \tilde{N}_1) \in \text{GL}(V^r/(\mathcal{K} + \mathcal{L}))^r$, where \tilde{N}_k is induced by the action of N_k on $V^r/(\mathcal{K} + \mathcal{L})$. The $K[F_r]$ -module $MC_\lambda(V) := (MC_\lambda(\mathbf{A}), V^r/(\mathcal{K} + \mathcal{L}))$ is called the *middle convolution* of \mathbf{M} with λ .

Theorem 2.2 Let $V = (\mathbf{M}, V) \in \text{Mod}(K[F_r])$, where $\mathbf{M} = (M_r, \dots, M_1) \in \text{GL}(V)^r$ and $\lambda \in K^\times$. Suppose that V has no 1-dimensional factors and/or submodules with the property that only one (or none) of the M_i act non-trivially.

(i) If $\lambda \neq 1$, then

$$\dim(MC_\lambda(V)) = \sum_{k=1}^r \text{rk}(M_k - 1) - (\dim(V) - \text{rk}(\lambda \cdot M_1 \cdots M_r - 1)).$$

(ii) If $\lambda_1, \lambda_2 \in K^\times$ such that $\lambda_1 \lambda_2 = \lambda$ and $(*)$ and $(**)$ hold for V , then

$$MC_{\lambda_2} MC_{\lambda_1}(V) \cong MC_\lambda(V).$$

(iii) Under the assumptions of (ii), if V is irreducible, then $MC_\lambda(V)$ is irreducible.

(iv) Let $\mathcal{B}_r = \langle \beta_1, \dots, \beta_{r-1} \rangle$ be the abstract Artin braid group, where the generators $\beta_1, \dots, \beta_{r-1}$ of \mathcal{B}_r satisfy the usual braid relations and act in the following way on tuples $(g_1, \dots, g_r) \in G^r$ (where G is a group):

$$\beta_i(g_r, \dots, g_1) = (g_r, \dots, g_{i-1}, g_i g_{i+1} g_i^{-1}, g_i, g_{i+2}, \dots, g_1), \quad i = 1, \dots, r-1.$$

For any $\beta \in \mathcal{B}_r$ there exists a $B \in \text{GL}(V^r / (\mathcal{K} + \mathcal{L}))$ such that

$$MC_\lambda(\beta(\mathbf{M})) = \beta(MC_\lambda(\mathbf{M}))^B,$$

where B acts via component-wise conjugation.

Proof: (i)-(iv) follow analogously to [3], Lemma 2.7, Lemma A.4, Theorem 3.5, Corollary 3.6 and Theorem 5.1 (in this order). \square

A Jordan block of eigenvalue $\alpha \in \bar{K}$ and of length l is denoted by $\mathbf{J}(\alpha, l)$. The following Lemma is a consequence of [3], Lemma 4.1:

Lemma 2.3 Let $V = (\mathbf{M}, V) \in \text{Mod}(K[F_r])$, where $\mathbf{M} = (M_1, \dots, M_r) \in \text{GL}(V)^r$ such that \mathbf{M} satisfies $(*)$ and $(**)$. Let $\lambda \in K^\times$ and $MC_\lambda(\mathbf{M}) = (\tilde{N}_1, \dots, \tilde{N}_r)$.

(i) Every Jordan block $\mathbf{J}(\alpha, l)$ occurring in the Jordan decomposition of M_i contributes a Jordan block $\mathbf{J}(\alpha\lambda, l')$ to the Jordan decomposition of \tilde{N}_i , where

$$l' := \begin{cases} l, & \text{if } \alpha \neq 1, \lambda^{-1}, \\ l-1 & \text{if } \alpha = 1, \\ l+1, & \text{if } \alpha = \lambda^{-1}. \end{cases}$$

The only other Jordan blocks which occur in the Jordan decomposition of \tilde{N}_i are blocks of the form $\mathbf{J}(1, 1)$.

(ii) Every Jordan block $\mathbf{J}(\alpha, l)$ occurring in the Jordan decomposition of M_{r+1} contributes a Jordan block $\mathbf{J}(\alpha\lambda, l')$ to the Jordan decomposition of \tilde{N}_{r+1} , where

$$l' := \begin{cases} l, & \text{if } \alpha \neq 1, \lambda^{-1}, \\ l+1 & \text{if } \alpha = 1, \\ l-1, & \text{if } \alpha = \lambda^{-1}. \end{cases}$$

The only other Jordan blocks which occur in the Jordan decomposition of \tilde{N}_{r+1} are blocks of the form $\mathbf{J}(\lambda, 1)$.

Remark 2.4 A geometric interpretation of the middle convolution can be given as follows, see [4]: Let $X = \mathbb{C} \setminus \{a_1, \dots, a_r\}$,

$$E = \{(x, y) \in \mathbb{C}^2 \mid x, y \neq a_i, i = 1, \dots, r, x \neq y\},$$

$p_i : E \rightarrow X$, $i = 1, 2$, be the i -th projection,

$$q : E \rightarrow \mathbb{C}^\times, (x, y) \mapsto y - x,$$

$j : E \rightarrow \mathbb{P}^1(\mathbb{C}) \times X$ the tautological inclusion and $\bar{p}_2 : \mathbb{P}^1(\mathbb{C}) \times X \rightarrow X$ the (second) projection onto X . Moreover, let \mathcal{L}_λ denote the Kummer sheaf associated to the representation, which sends a generator of $\pi_1(\mathbb{C}^\times)$ to λ . Let \mathcal{F} be a local system on X , corresponding to a tuple $\mathbf{M} := (M_r, \dots, M_1) \in \mathrm{GL}_m(\mathbb{C})^r$ (by a choice of a homotopy base) and let $\lambda \in \mathbb{C}^\times \setminus 1$. Then, under the assumptions of Thm. 2.2, $MC_\lambda(\mathbf{M})$ corresponds to the higher direct image sheaf

$$R^1(\bar{p}_2)_*(j_*(p_1^*(\mathcal{F}) \otimes q^*(\mathcal{L}_\lambda)))$$

(under the same choice of homotopy base as before).

3 Painlevé VI and the braid group orbits

It is well known, that PVI arises from isomonodromic deformations as follows, compare to [6] and [5]: Consider Fuchsian systems

$$(1) \quad \frac{d\Phi}{dz} = \left(\sum_{i=1}^3 \frac{A_i}{z - a_i} \right) \Phi,$$

where $A_i \in \mathrm{SL}_2(\mathbb{C})$. It is well known, that the isomonodromic deformations of the system (1) are described by the Schlesinger equations

$$(2) \quad \frac{\partial A_i}{\partial a_j} = \frac{[A_i, A_j]}{a_i - a_j} \quad \text{if } i \neq j, \quad \text{and} \quad \frac{\partial A_i}{\partial a_i} = - \sum_{j \neq i} \frac{[A_i, A_j]}{a_i - a_j}.$$

Set $A_4 := -(A_1 + A_2 + A_3)$ and let O_i denote the adjoint orbit, containing A_i . Let M be the quotient of

$$\left\{ (A_1, \dots, A_4) \in O_1 \times \dots \times O_4 \mid \sum A_i = 0 \right\}$$

modulo overall conjugation by $\mathrm{SL}_2(\mathbb{C})$, let $B := \mathbb{C}^3 \setminus \text{diagonals}$ and consider the trivial fibre bundle $\mathcal{M} := M \times B \rightarrow B$. Since the Schlesinger equations are invariant under overall conjugation, they give rise to the *isomonodromy connection* ∇ on the fibre bundle $\mathcal{M} \rightarrow B$.

For $\mathcal{C} := (C_1, \dots, C_4)$, $C_i := \exp(2\pi\sqrt{-1}O_i)$ and $(a_1, a_2, a_3) \in B$, consider the set $\mathrm{Hom}_{\mathcal{C}}(\pi_1(\mathbb{C} \setminus \{a_1, a_2, a_3\}), \mathrm{SL}_2(\mathbb{C}))/\mathrm{SL}_2(\mathbb{C})$ where a simple (counterclockwise) loop around a point a_i is mapped into C_i and a simple (clockwise) loop

around the set $\{a_1, a_2, a_3\}$ is mapped into C_4 . When (a_1, a_2, a_3) varies in B , these sets fit together to a fibre bundle $\psi : H \rightarrow B$. This fibre bundle has a complete flat connection Δ which is defined locally (say, $a_i \in D_i$, $i = 1, 2, 3$, where D_i is a small open disk with center a_i), by identifying representations which take the same values at simple loops γ_i around the disks D_i . For fixed $(a_1^o, a_2^o, a_3^o) \in B$ the set $\psi^{-1}(a_1^o, a_2^o, a_3^o)$ identifies with the set

$$\mathcal{N}(\mathcal{C}) := \{(g_1, \dots, g_4) \in C_1 \times \dots \times C_4 \mid g_1 \dots g_4 = 1\} / \mathrm{SL}_2(\mathbb{C}),$$

by fixing a homotopy base of $\pi_1(\mathbb{C} \setminus \{a_1^o, a_2^o, a_3^o\})$. Let $[g_1, \dots, g_4] \in \mathcal{N}(\mathcal{C})$ denote the equivalence class of (g_1, \dots, g_4) . By completeness, there is a unique global section of Δ through any point $P \in H$. Moreover, the analytic continuation of a section through $[g_1, \dots, g_4] \in \mathcal{N}(\mathcal{C})$ along a closed path γ at (a_1^o, a_2^o, a_3^o) is given by $[g_1, \dots, g_4]^\gamma$, where the (pure) braid group $B^3 := \pi_1(B, (a_1^o, a_2^o, a_3^o))$ acts as a subgroup of the full Artin braid group on tuples of matrices as in Thm. 2.2, (iv). Thus, sections of Δ with finite branching correspond to finite braid group orbits of the set $\mathcal{N}(\mathcal{C})$.

One has a natural map of fibre bundles

$$\nu : \mathcal{M} \longrightarrow H$$

which is induced by taking the monodromy of a Fuchsian system. By construction, the isomonodromy connection on \mathcal{M} is the pullback of ∇ . Thus, finite branching sections of ∇ correspond to finite braid group orbits on $\mathcal{N}(\mathcal{C})$.

The inclusion $\iota : \mathbb{C} \setminus \{0, 1\} \hookrightarrow B$, $t \mapsto (0, t, 1)$, induces an inclusion $\pi_1(\mathbb{C} \setminus \{0, 1\}) \rightarrow B^3$. Let P^3 be the image of this map. It is well known, that the Painlevé equation PVI arises from ∇ by restricting the singularity positions to $(0, t, 1)$, by a suitable choice of local coordinates of M and by a parameter elimination. Thus, finite branching solutions of PVI are obtained from finite branching solutions of ∇ , which correspond to finite braid group orbits on $\mathcal{N}(\mathcal{C})$. It is expected, that these solutions of PVI are always algebraic, see [1], P. 1012.

4 From triples pseudo-reflections in GL_3 to triples in SL_2

Let $G := \mathrm{SL}_2(\mathbb{C})$. Any triple $(M_3, M_2, M_1) \in G^3$ gives rise to a tuple $\mathbf{m} := \mathbf{m}(M_3, M_2, M_1) := (m_1, m_2, m_3, m_{13}, m_{23}, m_{13}, m_{321}) \in \mathbb{C}^7$, where

$$m_1 := \mathrm{Tr}(M_1), \quad m_2 := \mathrm{Tr}(M_2), \quad m_3 := \mathrm{Tr}(M_3),$$

$$m_{12} := \mathrm{Tr}(M_1 M_2), \quad m_{23} := \mathrm{Tr}(M_2 M_3), \quad m_{13} := \mathrm{Tr}(M_1 M_3)$$

and

$$m_{321} := \mathrm{Tr}(M_3 M_2 M_1).$$

Such seven-tuples satisfy the so-called *Fricke relation*, see [2], Section 2. Moreover, the induced braid group action can be explicitly determined, loc. cit..

Let $V := \mathbb{C}^3$, $e_i \in V$, $\alpha_i \in V^*$, $i = 1, 2, 3$, and let $r_i := 1 + e_i \otimes \alpha_i$, $i = 1, 2, 3$, be pseudo-reflections in $\mathrm{GL}_3(\mathbb{C})$. Choose complex numbers $n_1, n_2, n_3, t_1, t_2, t_3$ such that t_i is a choice of a square root of $\det(r_i)$, that the product $r_3 r_2 r_1$ has eigenvalues $\{n_1^2, n_2^2, n_3^2\}$ and that the square roots are chosen so that $t_1 t_2 t_3 = n_1 n_2 n_3$. Consider the following eight GL_3 -invariant functions on the space of triples of pseudo-reflections in GL_3 :

$$\begin{aligned} t_1^2, \quad t_2^2, \quad t_3^2 \\ t_{12} := \mathrm{Tr}(r_1 r_2) - 1, \quad t_{23} := \mathrm{Tr}(r_2 r_3) - 1, \quad t_{13} := \mathrm{Tr}(r_1 r_3) - 1, \\ t_{321} := n_1^2 + n_2^2 + n_3^2, \quad t'_{321} := (n_1 n_2)^2 + (n_2 n_3)^2 + (n_1 n_3)^2. \end{aligned}$$

Again, such eight-tuples satisfy certain relations and the braid group action can be explicitly determined (loc. cit.).

Now, it is shown in loc. cit. that the following map is invariant under the action of the braid group: Suppose that one is given a tuple

$$\mathbf{t} := \mathbf{t}(r_3, r_2, r_1) := (t_1, t_2, t_3, n_1, n_2, n_3, t_{12}, t_{23}, t_{13}),$$

associated to a triple of pseudo-reflections. Then one can define a map φ taking \mathbf{t} to some $\mathrm{SL}_2(\mathbb{C})$ data \mathbf{m} as follows:

$$\begin{aligned} m_1 &:= \frac{t_1}{n_1}, \quad m_2 := \frac{t_2}{n_1} + \frac{n_1}{t_2}, \quad m_3 := \frac{t_3}{n_1} + \frac{n_1}{t_3}, \\ (3) \quad m_{12} &:= \frac{t_{12}}{t_1 t_2}, \quad m_{23} := \frac{t_{23}}{t_2 t_3}, \quad m_{13} := \frac{t_{13}}{t_1 t_3}, \\ m_{321} &:= \frac{n_2}{n_3} + \frac{n_3}{n_2}. \end{aligned}$$

It is shown in loc. cit., Thm1., that this map is B_3 -invariant, thus finite B_3 -orbits of $\mathrm{SL}_2(\mathbb{C})$ -triples are obtained from triples of generators of three-dimensional complex reflection groups. These in turn lead to new algebraic solutions, of P_{VI} .

The following result shows, how the map φ is related to the middle convolution:

Theorem 4.1 *Let $\mathbf{r} := (r_3, r_2, r_1) \in \mathrm{GL}_3(\mathbb{C})$ be a triple of pseudo-reflections and let t_1, \dots, n_1, \dots and \mathbf{t} be as above. Suppose that $r_3 r_2 r_1$ has three distinct eigenvalues. Let $\lambda \in \mathbb{C}^\times$ be an eigenvalue of $r_3 r_2 r_1$,*

$$(M'_3, M'_2, M'_1) := MC_\lambda(\mathbf{r})$$

and

$$(M_3, M_2, M_1) := (n_3 M'_3, n_2 M'_2, n_1 M'_1).$$

Then

$$\mathbf{m}(M_3, M_2, M_1) = \varphi(\mathbf{t}).$$

Proof:

5 Some examples

□

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